J. Fluid Mech. (1991), *vol.* 233, *pp.* 157–163
Printed in Great Britain

On the linear stability of inviscid incompressible plane parallel flow

By E. M. BARSTON
Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, Chicago, IL 60680, USA

(Received 30 January 1990 and in revised form 2 January 1991)

A class of constants of motion is obtained for the time-dependent form of the Rayleigh stability equation, which provides stability criteria for velocity profiles with multiple inflexion points.

1. Introduction

The standard method for obtaining stability criteria from the linearized equations for an inviscid incompressible fluid in plane parallel flow is a normal-mode analysis, which leads to the Rayleigh stability equation (Drazin & Reid 1981, equation (21.17); Drazin & Howard 1966, equation (2.11)). Our approach is based on the time-dependent form of this equation, namely

$$\mathrm{P}\dot{\phi}+\mathrm{i}\alpha(U\mathrm{P}+U'')\,\phi=0, \quad a<z<b, \quad t>0, \tag{1.1}$$

with rigid-wall boundary conditions

$$\phi(a,t;\alpha)=0=\phi(b,t;\alpha), \quad t\geqslant 0. \tag{1.2}$$

Here α is the real wavenumber, $U(z)$ is the basic velocity profile, $\phi(z,t;\alpha)$ is the Fourier transform (with respect to the space variable in the direction of the basic flow) of the stream function for a small perturbation about the basic flow, P is the differential operator $-\partial^2/\partial z^2+\alpha^2$, $'\equiv \mathrm{d}/\mathrm{d}z$, and $\dot{}\equiv\partial/\partial t$. In this paper we develop criteria which guarantee that the solutions ϕ will be pointwise bounded independently of t. Our method permits results for profiles $U(z)$ with any finite number of inflexion points, and we recover the Rayleigh and Fjørtoft criteria (Drazin & Reid 1981; Drazin & Howard 1966) when U'' has no zeros or just one.

2. Stability

We assume that $U''(z)$ is continuous on $[a,b]$ and consider solutions ϕ of (1.1) that are, for each fixed $t>0$, twice continuously differentiable functions of z on $[a,b]$ and satisfy the boundary conditions (1.2). Then P admits the inverse P^{-1}, an integral operator whose kernel is the Green's function for P (Craik 1972, equation (2.1)). The operator P^{-1} is Hermitian and positive with respect to the complex inner product

$$(f,g)=\int_a^b \bar{f}(z)\,g(z)\,\mathrm{d}z, \quad \text{and} \quad \|\mathrm{P}^{-1}\|=\Lambda^{-1}$$

where $\Lambda=[\pi/(b-a)]^2+\alpha^2$ is the least eigenvalue of P. For every solution ϕ satisfying our smoothness conditions, we define

$$\xi(z,t;\alpha)\equiv \mathrm{P}\phi(z,t;\alpha), \quad a\leqslant z\leqslant b, \quad t\geqslant 0, \tag{2.1}$$

so that
$$\phi = \mathrm{P}^{-1}\xi \tag{2.2}$$
and (1.1) takes the form
$$\dot{\xi} = -\mathrm{i}\alpha W\xi, \quad t > 0 \quad \text{with} \quad W \equiv U''\mathrm{P}^{-1} + U. \tag{2.3}$$
Our stability analysis hinges upon the elementary observation that
$$[U'']^{-1}W = \mathrm{P}^{-1} + U/U''$$
is Hermitian as is $[U'']^{-1}$ (assuming that U'' has no zeros in $[a,b]$ so that the operator $[U'']^{-1}$ is defined). This implies that $(\mathrm{d}/\mathrm{d}t)(\xi,[U'']^{-1}\xi) \equiv 0$ for all solutions ξ of (2.3). Indeed, we have the following result (Barston 1977): If G is independent of t and G and GW are both Hermitian then so is GW^n for any positive integer n; and if $p(x)$ is any real polynomial in x, G$p(W)$ is Hermitian and $(\mathrm{d}/\mathrm{d}t)(\xi, \mathrm{G}p(W)\xi) \equiv 0$ for any solution ξ of (2.3), i.e. $(\xi, \mathrm{G}p(W)\xi)$ is a constant of the motion. Thus if G$p(W)$ is positive definite (i.e. $(f, \mathrm{G}p(W)f) \geqslant \delta\|f\|^2$ where $\delta > 0$ and δ is independent of f) we have
$$\delta\|\xi\|^2 \leqslant (\xi, \mathrm{G}p(W)\xi) = (\xi_0, \mathrm{G}p(W)\xi_0), \quad t \geqslant 0, \tag{2.4}$$
where $\xi_0 \equiv \xi(z,o;\alpha)$. Using the Cauchy–Schwarz inequality, with $a \leqslant z \leqslant b$ and $t \geqslant 0$,
$$\begin{aligned}|\phi(z,t;\alpha)|^2 &= \left|\int_a^z \phi'(x,t;\alpha)\,\mathrm{d}x\right|^2 \leqslant \left(\int_a^z |\phi'|^2\,\mathrm{d}x\right)\left(\int_a^z \mathrm{d}x\right)\\ &\leqslant (z-a)\left(\int_a^b |\phi'|^2\,\mathrm{d}x\right) \leqslant (z-a)(\phi, \mathrm{P}\phi) = (z-a)(\xi, \mathrm{P}^{-1}\xi)\\ &\leqslant (z-a)\,\|\mathrm{P}^{-1}\|(\xi,\xi) \leqslant (z-a)\,\delta^{-1}\Lambda^{-1}(\xi_0, \mathrm{G}p(W)\xi_0) \equiv R(z,\alpha),\end{aligned} \tag{2.5a}$$
which demonstrates the pointwise boundedness of ϕ independent of t. Thus we have stability for perturbations of wavenumber α. If G$p(W)$ is positive definite for all α, the inverse Fourier transform of $\phi(z,t;\alpha)$ leads to
$$|\hat{\phi}(y,z,t)| = \left|\int_{-\infty}^{\infty} \phi(z,t;\alpha)\,\mathrm{e}^{\mathrm{i}\alpha y}\,\mathrm{d}\alpha\right| \leqslant \int_{-\infty}^{\infty} |\phi|\,\mathrm{d}\alpha \leqslant \int_{-\infty}^{\infty} R^{\frac{1}{2}}(z,\alpha)\,\mathrm{d}\alpha, \tag{2.5b}$$
where y is the spatial coordinate in the direction of the basic flow and $\hat{\phi}(y,z,t)$ is the (untransformed) stream function. Hence $\hat{\phi}$ is bounded independently of t for all initial disturbances for which $\int_{-\infty}^{\infty} R^{\frac{1}{2}}\,\mathrm{d}\alpha$ converges.

We adopt this as our definition of stability, and seek to construct positive definite operators of the form G$p(W)$.

In the circumstance that U'' does not vanish on $[a,b]$, so that either U'' or $-U''$ is positive definite on $[a,b]$, we take G equal to the reciprocal of whichever is positive, let $p(x) = 1$, and conclude that (2.5a) holds with $\delta = \min_{[a,b]}|U''(z)|^{-1} > 0$. Thus we have Rayleigh's stability theorem: $U'' \neq 0$ on $[a,b]$ implies stability.

There remains the question of stability when U'' has at least one zero in $[a,b]$. It proves helpful to return briefly to the case $U'' \neq 0$. With $\mathrm{G} = [U'']^{-1}$ and C_k a real constant, we consider the following operators:
$$\mathrm{G}_1 \equiv \mathrm{G}(W - C_1) = \frac{U - C_1}{U''} + \mathrm{P}^{-1}, \tag{2.6}$$
$$\mathrm{G}_2 \equiv \mathrm{G}(W-C_1)(W-C_2) = \frac{(U-C_1)(U-C_2)}{U''} + (U-\hat{C}_2)\,\mathrm{P}^{-1} + \mathrm{P}^{-1}(U-\hat{C}_2) + \mathrm{P}^{-1}U''\mathrm{P}^{-1}, \tag{2.7}$$

$$\begin{aligned}\mathrm{G}_3 \equiv \mathrm{G}(W-C_1)(W-C_2)(W-C_3) = {} & \frac{(U-C_1)(U-C_2)(U-C_3)}{U''} \\ & +(U-\hat{C}_3)^2\mathrm{P}^{-1}+\mathrm{P}^{-1}(U-\hat{C}_3)^2+(U-\hat{C}_3)\,\mathrm{P}^{-1}(U-\hat{C}_3) \\ & +[\tfrac{3}{2}(\hat{C}_3)^2-\tfrac{1}{2}(C_1^2+C_2^2+C_3^2)]\,\mathrm{P}^{-1}+\mathrm{P}^{-1}\,U''\mathrm{P}^{-1}\,U''\mathrm{P}^{-1} \\ & +(U-\hat{C}_3)\,\mathrm{P}^{-1}\,U''\mathrm{P}^{-1}+\mathrm{P}^{-1}\,U''\mathrm{P}^{-1}(U-\hat{C}_3) \\ & +\mathrm{P}^{-1}(U-\hat{C}_3)\,U''\mathrm{P}^{-1}, \end{aligned} \tag{2.8}$$

where $\hat{C}_n \equiv (\sum_{j=1}^{n} C_j)/n$. In general,

$$\mathrm{G}_n \equiv \mathrm{G}(W-C_1)(W-C_2)\dots(W-C_n) = \frac{(U-C_1)(U-C_2)\dots(U-C_n)}{U''}+\mathrm{H}_n, \tag{2.9}$$

where H_n has no terms containing $(U'')^{-1}$ and is well-defined, bounded, and Hermitian regardless of whether U'' has zeros. Thus the right-hand sides of (2.6)–(2.9) will define G_n even in the case where U'' has zeros, provided that the real constants $C_1, \dots, C_n$ can be selected so that

$$(U-C_1)(U-C_2)\dots(U-C_n)/U''$$

is extendible to a continuous function on $[a,b]$. This requires that $(U-C_1)(U-C_2)\dots(U-C_n)$ vanish at each zero of U'' in $[a,b]$, and that for each such zero ζ,

$$\lim_{z\to\zeta}\frac{(U-C_1)(U-C_2)\dots(U-C_n)}{U''}$$

exists as a finite number. We assume henceforth that this holds and interpret $(U-C_1)(U-C_2)\dots(U-C_n)/U''$, wherever it appears, as the continuous extension of this function to all of $[a,b]$. We define G_n by the right-hand side of the appropriate equation from (2.6)–(2.9). Then G_n is Hermitian, and it can be verified directly that $\mathrm{G}_n W$ is also Hermitian, so that $(\mathrm{d}/\mathrm{d}t)(\xi, \mathrm{G}_n \xi) \equiv 0$ for solutions ξ of (2.3). Sufficient conditions for the positive definiteness of a G_n then become, via (2.4) and (2.5), sufficient conditions for stability.

3. G_1 and Fjørtoft's criterion

Suppose that U'' has zeros in $[a,b]$ and that U at every zero of U'' has the common value C_1. We assume that $\lim_{z\to\zeta}(U-C_1)/U''$ exists and is finite for every zero ζ of U'' in $[a,b]$.

Let $(U-C_1)/U'' \geqslant 0$ for all z in $[a,b]$. Since $(\xi, \mathrm{G}_1\xi)$ is a constant of the motion for all solutions ξ of (2.3), we have

$$(\xi, \mathrm{P}^{-1}\xi) \leqslant (\xi, \mathrm{G}_1\xi) = (\xi_0, \mathrm{G}_1\xi_0), t \geqslant 0,$$

and (2.5) yields $|\phi(z,t;\alpha)| \leqslant (z-a)^{\frac{1}{2}}(\xi_0, \mathrm{G}_1\xi_0)^{\frac{1}{2}}, t \geqslant 0$, and we have stability. This is Fjørtoft's criterion.

On the other hand, if $M \equiv \max_{[a,b]}(U-C_1)/U'' < -\|\mathrm{P}^{-1}\| = -\Lambda^{-1}$, then

$$(f, \mathrm{G}_1 f) = (f, [(U-C_1)/U'']f)+(f, \mathrm{P}^{-1}f) \leqslant (M+\|\mathrm{P}^{-1}\|)\,\|f\|^2 < 0,$$

so that $-\mathrm{G}_1$ is positive definite and we have stability for disturbances with wavenumber α satisfying $\alpha^2 > -M^{-1}-[\pi/(b-a)]^2$. This is criterion (v)′ of Craik (1972). Thus if $M < -(b-a)^2/\pi^2$, stability obtains for every α and the flow is stable.

Suppose $\mu(z)$ is a basic flow with a single inflexion point at z_1, with

$$\mu''(z) = (z-z_1)\rho(z), \rho(z) > 0 \quad \text{on } [a,b].$$

Consider the two-parameter family of flows $U(z) = \mu(z)+Az+\gamma$ (A, γ real constants) with the same second derivative $\mu''(z)$. Let

$$\mu_1(z) \equiv [\mu(z)-\mu(z_1)]/(z-z_1) \quad \text{for} \quad z \neq z_1, \mu_1(z_1) \equiv \mu'(z_1).$$

Then $\mu_1(z)$ is continuous on $[a,b]$, $\mu(z)-\mu(z_1) = (z-z_1)\mu_1(z)$, and the choice

$$C_1 = U(z_1) = \mu(z_1)+Az_1+\gamma \quad \text{gives} \quad \mathrm{G}_1 = (\mu_1(z)+A)/\rho+\mathrm{P}.$$

Hence $\mathrm{G}_1 > 0$ if $A > A_2 \equiv -\min_{[a,b]}\mu_1(z)$ while

$$-\mathrm{G}_1 > 0 \quad \text{if} \quad A < A_1 \equiv -\max_{[a,b]}[(b-a)^2/\pi^2)\rho(z)+\mu_1(z)].$$

Thus the addition of a 'background' shear flow Az to μ results in a stable flow provided that the shear coefficient A lies outside the interval $[A_1, A_2]$. We shall see that a similar result holds for basic flows with any finite number of inflexion points.

4. The case of two or more zeros

We consider the case where U'' has two distinct simple zeros $z_1 < z_2$ in (a,b). Specifically, we take $U''(z) = (z-z_1)(z-z_2)\rho(z)$, where $\rho(z)$ is a non-vanishing continuous function on $[a,b]$. Then for $C_j = U(z_j), j = 1,2$,

$$\lim_{z\to z_j}\frac{(U-C_1)(U-C_2)}{U''} = \frac{U(z_2)-U(z_1)}{(z_2-z_1)\rho(z_j)}U'(z_j)$$

by L'Hôpital's rule, and $\mathrm{G}_2 = [(U-C_1)(U-C_2)]/U''+\mathrm{H}_2$ is well-defined by the right-hand side of (2.7). If H_2 were positive definite, we would have a simple stability criterion analogous to the Fjørtoft criterion, but unfortunately H_2 is indefinite. Indeed, by (2.1)–(2.2),

$$\begin{aligned}(\xi, \mathrm{H}_2\xi) &= (\xi, [(U-\hat{C}_2)\,\mathrm{P}^{-1}+\mathrm{P}^{-1}(U-\hat{C}_2)+\mathrm{P}^{-1}U''\mathrm{P}^{-1}]\,\xi) \\ &= (\phi, [\mathrm{P}(U-\hat{C}_2)+(U-\hat{C}_2)\,\mathrm{P}+U'']\,\phi) \\ &= 2\int_a^b (U-\hat{C}_2)(|\phi'|^2+\alpha^2|\phi|^2)\,\mathrm{d}z, \end{aligned} \tag{4.1}$$

where we have used the fact that

$$\mathrm{P}U+U\mathrm{P}+U'' = 2\left[-\frac{\partial}{\partial z}U\frac{\partial}{\partial z}+\alpha^2 U\right].$$

The continuity of U implies that

$$\min_{[a,b]} U \leqslant \hat{C}_2 = \tfrac{1}{2}[U(z_1)+U(z_2)] \leqslant \max_{[a,b]} U,$$

and since U is smooth and z_1 and z_2 are points of inflexion, $U(z_j)$ cannot be the maximum or the minimum value of U on $[a,b]$. Thus $\min_{[a,b]}U < \hat{C}_2 < \max_{[a,b]}U$, and it follows immediately from (4.1) that H_2 is indefinite. However, since

$$(\xi, \mathrm{P}^{-1}\xi) = (\phi, \mathrm{P}\phi) = \int_a^b [|\phi'|^2+\alpha^2|\phi|^2]\,\mathrm{d}z,$$

$$\left(\min_{[a,b]} U-\hat{C}_2\right)(\xi, \mathrm{P}^{-1}\xi) \leqslant \int_a^b (U-\hat{C}_2)[|\phi'|^2+\alpha^2|\phi|^2]\,\mathrm{d}z \leqslant \left(\max_{[a,b]} U-\hat{C}_2\right)(\xi, \mathrm{P}^{-1}\xi),$$

and (4.1) yields

$$2\left(\min_{[a,b]} U-\hat{C}_2\right)\Lambda^{-1} \leqslant \mathrm{H}_2 \leqslant 2\left(\max_{[a,b]} U-\hat{C}_2\right)\Lambda^{-1}. \tag{4.2}$$

Let

$$m_2 \equiv \min_{[a,b]} \frac{[U(z)-U(z_1)][U(z)-U(z_2)]}{U''(z)}, \quad M_2 \equiv \max_{[a,b]} \frac{[U(z)-U(z_1)][U(z)-U(z_2)]}{U''(z)}.$$

Then

$$m_2+2\left(\min_{[a,b]} U-\hat{C}_2\right)\Lambda^{-1} \leqslant \mathrm{G}_2 \leqslant M_2+2\left(\max_{[a,b]} U-\hat{C}_2\right)\Lambda^{-1},$$

so that if $m_2 > 0$, G_2 will be positive definite for disturbances with wavenumber α satisfying

$$\alpha^2 > 2m_2^{-1}\left(\hat{C}_2-\min_{[a,b]} U\right)-\pi^2(b-a)^{-2},$$

while if $M_2 < 0$, $-\mathrm{G}_2$ will be positive definite for disturbances with wavenumber α satisfying

$$\alpha^2 > -2M_2^{-1}\left(\max_{[a,b]} U-\hat{C}_2\right)-\pi^2(b-a)^{-2}.$$

Thus if

$$m_2 > 2\pi^{-2}(b-a)^2\left(\hat{C}_2-\min_{[a,b]} U\right) \tag{4.3}$$

or

$$M_2 < -2\pi^{-2}(b-a)^2\left(\max_{[a,b]} U-\hat{C}_2\right) \tag{4.4}$$

stability obtains for every α and the flow is stable.

As an elementary example of the application of these inequalities, consider the basic flow

$$U(z) = V\left[\left(\frac{z}{b}\right)^4-\frac{4}{3}\left(\frac{z}{b}\right)^3+A\left(\frac{z}{b}\right)\right], \quad -b \leqslant z \leqslant b,$$

where V and A are constants and the inflexion points are $z_1 = 0$ and $z_2 = \frac{2}{3}b$. It is readily demonstrated, with a little algebra and some simple (and rough) estimates, that (4.3) holds for $A < -7$, and so this flow is stable whenever $A < -7$.

Suppose $\mu(z)$ is a basic flow with $\mu''(z) = (z-z_1)(z-z_2)\rho(z)$ with $\rho(z)$ positive and continuous on $[a,b]$, and consider the family of flows $U(z) = \mu(z)+Az+\gamma$ (A, γ constants).

We define

$$\mu_j(z) \equiv \begin{cases} \dfrac{\mu(z)-\mu(z_j)}{z-z_j}, & z \neq z_j, a \leqslant z \leqslant b \quad (j = 1,2). \\ \mu'(z_j), & z = z_j \end{cases}$$

Then μ_j is continuous on $[a,b]$ and $\mu(z)-\mu(z_j) = (z-z_j)\,\mu_j(z)$ for $a \leqslant z \leqslant b$. Hence G_2, defined by the right-hand side of (2.7) with $C_j = U(z_j), j = 1,2$, takes the form

$$\begin{aligned}\mathrm{G}_2 = {} & \frac{(\mu_1+A)(\mu_2+A)}{\rho}+A[(z-\hat{z})\,\mathrm{P}^{-1}+\mathrm{P}^{-1}(z-\hat{z})]+(\mu-\hat{\mu})\,\mathrm{P}^{-1} \\ & +\mathrm{P}^{-1}(\mu-\hat{\mu})+\mathrm{P}^{-1}\mu''\,\mathrm{P}^{-1},\end{aligned} \tag{4.5}$$

where $\hat{z} = \frac{1}{2}(z_1+z_2), \hat{\mu} = \frac{1}{2}[\mu(z_1)+\mu(z_2)]$.

Since $\min_{[a,b]}(\mu_1+A)(\mu_2+A)/\rho$ can be made as large as we please by taking $|A|$ sufficiently large and is of order A^2, while the remaining terms consist of bounded operators whose norms are at most of order A, G_2 will be positive definite for $|A|$ sufficient large. Specifically, suppose

$$A > A_+ \equiv -\min\left\{\min_{[a,b]}\mu_1, \min_{[a,b]}\mu_2\right\}.$$

Then $(A+\mu_1)(A+\mu_2)/\rho \geqslant (A-A_+)^2/\tilde{\rho}$, where $\tilde{\rho} = \max_{[a,b]}\rho$. If U is replaced by z and then by μ in (4.1) and (4.2), the results obtained together with (4.5) yield, for $A > A_+$,

$$\mathrm{G}_2 \geqslant [(A-A_+)^2-(A-A_+)B-C](\tilde{\rho})^{-1}, \tag{4.6}$$

where

$$B = 2\tilde{\rho}\Lambda^{-1}(\hat{z}-a) > 0, \quad \text{and} \quad C = 2\tilde{\rho}\Lambda^{-1}[\hat{\mu}+A_+\hat{z}-\min_{[a,b]}(\mu+A_+z)] > 0.$$

The inequality (4.6) shows that G_2 is positive definite when

$$A > A_+ + \tfrac{1}{2}[B+(B^2+4C)^{\frac{1}{2}}], \tag{4.7}$$

and we have stability for perturbations of wavenumber α. Similarly, it can be shown that G_2 will again be positive definite if

$$A < A_- - \tfrac{1}{2}[D+(D^2+4E)^{\frac{1}{2}}], \tag{4.8}$$

where

$$A_- = -\max\left\{\max_{[a,b]}\mu_1, \max_{[a,b]}\mu_2\right\}, \quad D = 2\tilde{\rho}\Lambda^{-1}(b-\hat{z}),$$

$$E = 2\tilde{\rho}\Lambda^{-1}[\hat{\mu}+A_-\hat{z}-\min_{[a,b]}(\mu+A_-z)].$$

Since $\Lambda^{-1} = [\pi^2(b-a)^{-2}+\alpha^2]^{-1} \leqslant (b-a)^2\pi^{-2}$, inequalities (4.7) and (4.8) will imply stability of the flow (i.e. stability for all α) provided that Λ^{-1} is replaced by $(b-a)^2\pi^{-2}$ in the definitions of B, C, D and E.

The symmetric flow $\mu = V\cos(\beta z/b)$, $-b \leqslant z \leqslant b$, with V a positive constant and $\frac{1}{2}\pi < \beta < \frac{3}{2}\pi$ has the two inflexion points $z_1 = -z_2 = -b\zeta_0$ where $\frac{1}{3} < \zeta_0 \equiv \pi(2\beta)^{-1} < 1$. One readily finds that

$$A_+ = -A_- = \frac{\beta V}{b}, \quad \hat{z} = 0 = \hat{\mu}, \quad \tilde{\rho} = 4V\beta^4\pi^{-2}b^{-4},$$

$$\min_{[-b,b]}(\mu+A_+z) = V(\cos\beta-\beta) = \min_{[-b,b]}(\mu+A_-z).$$

The replacement of Λ^{-1} by $(b-a)^2\pi^{-2} = 4b^2\pi^{-2}$ gives

$$B = D = 2\zeta_0^{-4}V/b, C = E = 2\zeta_0^{-4}V^2b^{-2}(\beta-\cos\beta),$$

so that by (4.7)–(4.8), the basic flow $U = V\cos(\beta z/b)+Az+\gamma, \frac{1}{2}\pi < \beta < \frac{3}{2}\pi$, will be stable for all A lying outside the interval $[-A_0, A_0]$, where

$$A_0 = A_1 + Vb^{-1}\zeta_0^{-4}\{1+[1+2\zeta_0^4(\beta-\cos\beta)]^{\frac{1}{2}}\} > A_1+2Vb^{-1}\zeta_0^{-4} \tag{4.9}$$

and $A_1 \equiv \beta V/b = \max_{[-b,b]}|\mu'(z)|$. As β increases from $\frac{1}{2}\pi$ to $\frac{3}{2}\pi$, z_2 decreases from b to $\frac{1}{3}b$ and ζ_0^{-4} increases from 1 to 81, so that A_0 increases away from A_1 dramatically. Note that the flow is stable for $0 < \beta < \frac{1}{2}\pi$ by Rayleigh's criterion.

In the case of n simple zeros, we write

$$U = \mu+Az+\gamma, \quad \text{where} \quad U'' = \mu'' = (z-z_1)\dots(z-z_n)\rho(z)$$

on $[a,b]$ with $a < z_1 < z_2 < \ldots < z_n < b$, define μ_j and $C_j = U(z_j)$ as before for $1 \leqslant j \leqslant n$, to obtain

$$(U-C_1)(U-C_2)\ldots(U-C_n)/U'' = (\mu_1+A)(\mu_2+A)\ldots(\mu_n+A)/\rho.$$

The minimum of the absolute value of this quantity on $[a,b]$ will become arbitrarily large for $|A|$ sufficiently large and is of order $|A^n|$, while the terms in H_n will be at most of order A^{n-1}, so that G_n or $-\mathrm{G}_n$ will be positive definite for $|A|$ sufficiently large. Thus if the absolute value of the shear coefficient A is sufficiently large, the flow will be stable.

5. Summary and conclusions

For a given basic flow $U(z)$, we have constructed a class of Hermitian operators G_n (equations (2.6)–(2.9)) which have the property that $(\mathrm{P}\phi, G_n \mathrm{P}\phi)$ is a constant of the motion for perturbations ϕ. The construction of a G_n which is positive (or negative) definite for all wavenumbers α implies that the stream function $\hat{\phi}(y,z,t)$ is bounded independently of t (equation $(2.5a,b)$) and hence that the flow is stable, i.e. perturbations which are initially small remain small for all $t > 0$. This method thus provides more information than the usual normal-mode approach, which, owing to the presence of a continuous spectrum for the Rayleigh equation, must be supplemented with an analysis of the initial-value problem (e.g. via the Laplace transform) to conclude stability (Drazin & Reid 1981).

The search for a positive (or negative) definite G_n led us to the Rayleigh criterion for flows with no inflexion points, to the Fjørtoft criterion for flows with one inflexion point, and to the stability criteria of (4.3) and (4.4) for flows with two inflexion points. Finally, it was shown that for flows with any finite number of inflexion points, the addition of a 'background' shear flow Az to the original flow will result in a stable flow if $|A|$ is sufficiently large, provided that U'' is the product of a polynomial in z and a continuous non-vanishing function $\rho(z)$.

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